Physics 212 Exam 3 Review Notes

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Contents

1 AC Circuits

1.1 LRC Basics

In an AC circuit, the current and the voltages across the various circuit elements all oscillate sinusoidally at some frequency ω (rads/s). (Note that there is also a frequency f which is measured in Hz, or cycles/s. f is related to ω by $\omega = 2\pi f$, and the period of the oscillation is $T = 1/f$.) At any time, the current through the circuit will be:

$$I(t) = I_{\max}\sin(\omega t + \theta_0) \tag{1}$$

where $I_{\max}\sin(\theta_0)$ is the initial ($t = 0$) value of the current.

For a series circuit (the only kind we deal with), the voltages across the various circuit elements will be:

$$\begin{aligned} V_C(t) &= \frac{Q(t)}{C} = \frac{1}{C}\int I(t)dt = \frac{1}{\omega C}I_{\max}\cos(\omega t + \theta_0) = -V_{C,\max}\cos(\omega t + \theta_0) && (2)\\ V_R(t) &= RI(t) = RI_{\max}\sin(\omega t + \theta_0) = V_{R,\max}\sin(\omega t + \theta_0) && (3)\\ V_L(t) &= L\frac{dI(t)}{dt} = \omega L I_{\max}\cos(\omega t + \theta_0) = V_{L,\max}\cos(\omega t + \theta_0) && (4)\end{aligned}$$

The factors $\frac{1}{\omega C} = X_C$ and $\omega L = X_L$ are called the capacitative and inductive reactances. They have units of resistance, and when we are discussing the maximum voltages across AC circuit components, we can use the following Ohm's law-like equations:

$$\begin{aligned} V_{C,\max} &= X_C I_{\max} && (5)\\ V_{R,\max} &= R I_{\max} && (6)\\ V_{L,\max} &= X_L I_{\max} && (7)\end{aligned}$$

The overall circuit has a parameter called the "impedance", Z, which, for an LRC circuit is:

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \tag{8}$$

If there is an EMF (a wall outlet, or whatever) driving the circuit then the voltage across the EMF will be:

$$\mathcal{E}(t) = \mathcal{E}_{\max}\sin(\omega t + \theta_0 + \phi) \tag{9}$$

where ϕ is the phase difference between the EMF and the current.

The maximum value of the driving EMF is related to the maximum value of the current flowing through the circuit by the Ohm-like relation:

$$\mathcal{E}_{\max} = Z I_{\max} \tag{10}$$

while the phase between the EMF and the current is:

$$\tan(\phi) = \frac{X_L - X_C}{R} = \frac{\omega L - \frac{1}{\omega C}}{R} \tag{11}$$

At any time, it must be true that:

$$\mathcal{E}(t) = V_C(t) + V_R(t) + V_L(t) \tag{12}$$

This is true only of the values at a particular time, not of the maximum values of the various voltages and currents.

1.2 Phasors

The voltages across all the circuit elements can be represented on a phasor diagram. The idea of a phasor diagram is that, for each voltage, you draw a vector whose length is the maximum possible value of that voltage and whose y component is equal to the value of that voltage *at the particular time you've chosen to draw the diagram.* If you take a phasor diagram drawn at a particular time and let time proceed forward, then all of these vectors (the phasors) will rotate about the origin counterclockwise, completing one full rotation in a time $T = 2\pi/\omega$.

When you draw a phasor diagram, you will notice that the phasor for the inductor is always 90° counterclockwise from ("ahead of") the phasor for the resistor, while the phasor for the capacitor is always 90° behind the resistor. The phasor for the EMF will have a variable angle ($-90^\circ < \phi < 90^\circ$) with respect to the resistor phasor. If the EMF phasor is counterclockwise from the resistor phasor ($\phi > 0$) then the EMF is said to lead the current (and the resistor voltage). If $\phi < 0$, the EMF is said to lag. If the EMF is exactly on top of the resistor phasor, then the circuit is in resonance ($\phi = 0$). Finally, note that the sum of the vectors for the resistor, capacitor, and inductor *must* be equal to the EMF phasor.

$$\vec{\mathcal{E}}(t) = \vec{V}_C(t) + \vec{V}_R(t) + \vec{V}_L(t) \tag{13}$$

Also, note that, looking at a phasor diagram, it can be observed that:

$$\begin{aligned} V_{R,\max} &= \mathcal{E}_{\max}\cos(\phi) && (14)\\ V_{L,\max} - V_{C,\max} &= \mathcal{E}_{\max}\sin(\phi) && (15) \end{aligned}$$

And further, using the relations between max current and the max voltages on the circuit elements, we can see that:

$$\begin{aligned} R &= Z\cos(\phi) && (16)\\ X_L - X_C &= Z\sin(\phi) && (17) \end{aligned}$$

1.3 RMS Values

The RMS value of any function $f(x)$ is the square root of the average value of $(f(x))^2$. (Thus, the Root Mean Square: we square it, take the mean, and take the square root.) In particular, for a sinusoidal (sine or cosine) function, the RMS value is always $1/\sqrt(2)$ times the maximum value. Since all of the voltages and the current in this problem are sinusoidal, we have the following relations:

$$\mathcal{E}_{\text{RMS}} = \frac{\mathcal{E}_{\max}}{\sqrt{2}} \tag{18}$$

$$
\begin{aligned}
I_{\mathrm{RMS}} &= \frac{I_{\mathrm{max}}}{\sqrt{2}} && (19)\\
V_{C,\mathrm{RMS}} &= \frac{V_{C,\mathrm{max}}}{\sqrt{2}} && (20)\\
V_{R,\mathrm{RMS}} &= \frac{V_{R,\mathrm{max}}}{\sqrt{2}} && (21)\\
V_{L,\mathrm{RMS}} &= \frac{V_{L,\mathrm{max}}}{\sqrt{2}} && (22)
\end{aligned}
$$

Note that the RMS value is always smaller than the maximum value, and it had darn well better be. (The average should not be greater than the maximum for *any* function.)

Further, be aware that the RMS voltages and current are related to each other in the exact same way as the maximum voltages and current, since we can derive an equation relating RMS values simply by multiplying both sides of an equation by $\frac{1}{\sqrt{2}}$. For example:

$$
\begin{aligned}
\mathcal{E}_{\mathrm{RMS}} &= Z I_{\mathrm{RMS}} && (23)\\
V_{R,\mathrm{RMS}} &= \mathcal{E}_{\mathrm{RMS}} \cos(\phi) && (24)\\
V_{L,\mathrm{RMS}} - V_{C,\mathrm{RMS}} &= \mathcal{E}_{\mathrm{RMS}} \sin(\phi) && (25)
\end{aligned}
$$

1.4 Power Dissipation and Energy Storage

All of the power dissipated by an LRC circuit is dissipated in the resistor. The instantaneous power dissipation by the resistor will just follow the usual formula, $P(t) = V_R(t)I(t) = R(I(t))^2$. But usually we are interested in the time-averaged power dissipation. A time-averaged quantity is written surrounded by angle brackets $\langle\cdot\rangle$. Since we already know that for a sinusoidal function, $f_{\mathrm{RMS}} = \sqrt{\langle (f(t))^2\rangle} = f_{\mathrm{max}}/\sqrt{2}$, then we have:

$$
\begin{aligned}
\langle P(t)\rangle &= \left\langle R(I(t))^2 \right\rangle = R\left\langle (I(t))^2 \right\rangle && (26)\\
&= R I_{\mathrm{RMS}}^2 = \frac{1}{2} R I_{\mathrm{max}}^2 && (27)\\
&= V_{R,\mathrm{RMS}} I_{\mathrm{RMS}} = \frac{1}{2} V_{R,\mathrm{max}} I_{\mathrm{max}} && (28)\\
&= \frac{V_{R,\mathrm{RMS}}^2}{R} = \frac{V_{R,\mathrm{max}}^2}{2R} && (29)\\
&= \frac{\mathcal{E}_{\mathrm{RMS}}^2 \cos^2(\phi)}{R} = \frac{\mathcal{E}_{\mathrm{max}}^2 \cos^2(\phi)}{2R} && (30)\\
&= \mathcal{E}_{\mathrm{RMS}} I_{\mathrm{RMS}} \cos(\phi) = \frac{1}{2}\mathcal{E}_{\mathrm{max}} I_{\mathrm{max}} \cos(\phi) && (31)
\end{aligned}
$$

and so forth.

The inductor and capacitor will have stored energy which oscillates over time depending on the current passing through the inductor and the charge stored in the capacitor. The inductor's energy will be maximized when the current is at its maximum (negative or positive) value. The capacitor's energy will be maximized when the charge stored in it is at its maximum negative or positive value. Note that, since $Q(t) = \int I(t)dt$, the capacitor will have maximum energy when the inductor has zero energy, and vice versa. Also note that, since both the current and charge have one maximum positive value and one maximum negative value in each full cycle of the circuit, the stored energy actually oscillates back and forth between these two elements at *twice* the oscillation frequency ω of the current, charge, and voltages.

1.5 Resonance

A circuit in resonance has a number of unique characteristics. The *definition* of resonance is that the current output is maximized for a given EMF. Looking at the above relation between maximum EMF and maximum current, we see that maximizing current output while holding the max EMF constant requires minimizing the impedance Z. The impedance will be minimized when $X_L = X_C$. Using this fact, we can derive all the other properties of resonance. At resonance:

$$\begin{aligned} X_L &= X_C && (32) \\ \omega = \omega_0 &= \frac{1}{\sqrt{LC}} && (33) \\ \phi &= 0 && (34) \\ Z &= R && (35) \\ V_{C,\text{max}} &= V_{L,\text{max}} && (36) \\ \mathcal{E}_{\text{max}} &= V_{R,\text{max}} && (37) \end{aligned}$$

ω_0 is called the resonant frequency of the circuit. If $\omega > \omega_0$ then $X_L > X_C$ and $\phi > 0$. If $\omega < \omega_0$ then $X_L < X_C$ and $\phi < 0$.

1.6 Quality (The Q-Factor)

If we draw a graph of the maximum current in the circuit as a function of the driving frequency ω, then the graph will show a peak at $\omega = \omega_0$. Imagine that ω_0 happens to be the frequency of a radio station we would like to pick up using the circuit, and that there is another radio station broadcasting at a frequency very near ω_0. We would like to design our circuit such that the station at ω_0 (acting as a driving EMF with frequency ω_0) creates a large current in our circuit, while the nearby station creates only a very small current. In other words, if our circuit is of high quality, then the response peak will drop off very sharply to either side of the resonant frequency, so that the other station would have to be practically on top of ω_0 to generate any response at all.

We measure the "quality" of our circuit by finding the two frequencies to either side of ω_0 for which the response is half that of the response at ω_0. The difference between these two frequencies is called the *full width at half maximum* (FWHM), and the ratio of ω_0 to the FWHM is called the quality (Q) factor. A narrower peak means a higher Q, so a higher Q means a higher quality circuit. Here are some formulas for Q:

$$Q = \frac{\omega_0 L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{R\omega_0 C} = \frac{X_{L,0}}{R} = \frac{X_{C,0}}{R} \approx \frac{\omega_0}{\text{FWHM}} \tag{38}$$

where $X_{L,0}$ and $X_{C,0}$ are the reactances at resonance ($\omega = \omega_0$).

1.7 LC Circuits

An LC circuit has no driving EMF, so it can only oscillate at its resonant frequency. Typically, you will start with the capacitor having been charged up (perhaps by an external DC RC circuit) and the inductor having no current flowing through it. So the total stored energy in the circuit will then be just the capacitor energy $U = U_L = Q^2/2C$. Then you close a switch, and the energy starts flowing back and forth between the inductor and capacitor. (The total energy remains constant, since there is no resistor in the circuit to dissipate it. If we added a resistor, the energy would still oscillate between the two at the same rate, but the total energy would decrease over time at a rate determined by the power dissipated in the resistor.)

Specifically, the capacitor will have a total charge:

$$Q(t) = Q_{\max} \cos(\omega_0 t) \tag{39}$$

while the current flowing through the inductor will be:

$$I(t) = I_{\max} \sin(\omega_0 t) \tag{40}$$

Note that it takes 1/4 cycle for the capacitor to become uncharged and the inductor to become fully charged.

If the instead inductor starts out charged and the capacitor starts uncharged, then the inductor will have a cosine dependence and the capacitor will have a sine dependence. Again, it will take 1/4 cycle for their roles to be reversed. Note that, as in the LRC circuit, the energy on either circuit element oscillates at *twice* the resonant frequency.

1.8 LR and RC Circuits

Typically, these circuits will be driven by an EMF, since if they were not, the resistor would quickly dissipate all the energy in the circuit. These circuits may be handled in exactly the same way as an LRC circuit with the exception that for an LR circuit, $X_C = 0$ and for an RC circuit, $X_L = 0$.

Also, note that neither an LR nor an RC circuit can reach resonance, since the phase will be $\tan(\phi) = X_L/R > 0$ for LR and $\tan(\phi) = -X_C/R < 0$ for RC.

1.9 Transformers

A transformer is an object made of two inductive coils (solenoids) wrapped around the same metal core. The metal core traps the magnetic field lines and causes the magnetic field inside both inductors to be the same, and the areas of the loops in each coil are also the same. So, recalling that magnetic flux is $\Phi_B = \vec{A} \cdot \vec{B}$, we see that if one coil has N_1 turns and the other has N_2 turns, then the fluxes through the two coils are related by:

$$\frac{\Phi_1}{\Phi_2} = \frac{N_1}{N_2} \tag{41}$$

Since the flux through an inductor is also $\Phi_B = LI$ and the voltage across an inductor is $V = L\,dI/dt = d\Phi_B/dt$, we thus see that the voltages across the two inductors are related by:

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} \tag{42}$$

Typically, the inductor on one side is hooked up to a circuit which contains a battery or AC voltage. This circuit will cause a voltage to be generated across the first inductor, which will cause, via the transformer, a proportionate voltage across the second inductor. The second inductor then acts as a power supply (battery or AC EMF) for the second circuit. So all you need to do is find the voltage across the first inductor by solving the first circuit, use the transformer equation to find the voltage across the second inductor, and then redraw the second inductor's circuit with that inductor replaced by a battery or AC generator with the same voltage.

If the power source is DC, then you simply set $V_2 = N_2 V_1/N_1$ and compute the second circuit as a standard DC circuit. If the power source is AC, then you set $V_{2,\max} = N_2 V_{1,\max}/N_1$, $\omega_2 = \omega_1$ and $\phi_2 = \phi_1$ for the second circuit's generator, and compute it as a standard LRC circuit.

Any power dissipated in the second circuit *comes from* the generator or battery driving the first circuit. There may be resistors in the first circuit which dissipate additional power, but the second circuit can *never* dissipate more power than is provided by the generator or battery.

The *voltage* driving the second circuit may be higher than the voltage driving the first circuit. This occurs if $N_2 > N_1$.

2 Electromagnetic Waves and Polarization

2.1 Basics

An electromagnetic plane wave consists of oscillating magnetic and electric fields vectors which are perpendicular to each other at every instant and "propagate" in a direction perpendicular to the direction in which they point. At any time, the length of the electric and magnetic field vectors are related by:

$$E(t) = cB(t) \tag{43}$$

and the maximum values of the two fields are related in the same way. The fields oscillate at the same frequency $\omega = 2\pi f$, and the wavelength (spatial distance between peaks) of their oscillations is $\lambda = v/f$, where v is the propagation velocity of the waves. In a vacuum or air, $v = c$. In a medium with refractive index n, $v = c/n$. Other characteristic parameters are the wavenumber $k = 2\pi/\lambda$ (measured in rads/m) and the period of the oscillation, $T = 1/f = 2\pi/\omega$. So:

$$v = \frac{c}{n} = \lambda f = \frac{\omega}{k} = \frac{\lambda}{T} \tag{44}$$

Note that the *frequency* of a plane wave in a medium of index n will be the same as its frequency in vacuum: $f_n = f$. But its *wavelength* and *speed* will be different from their vacuum values: $\lambda_n = \lambda/n$, $v_n = c/n$.

Also, be aware that when we talk about plane electromagnetic waves, we are talking about *light*, which is why their propagation speed in vacuum is the same as the vacuum speed of light, $c = 3 \cdot 10^8$ m/s. So all of the above equations are also applicable when we are discussing light.

A basic, linearly polarized electromagnetic wave might be written as follows:

$$\vec{E}(t) = E_0 \sin(kz - \omega t)\hat{x} \tag{45}$$

$$\vec{B}(t) = B_0 \sin(kz - \omega t)\hat{y} \tag{46}$$

The electric field vector consists of several pieces:

First, it has an amplitude E_0 which (in this particular case) is the maximum length of the electric field vector. The amplitude of the magnetic field vector will be $1/c$ times this.

Next, it has a sine factor. (This may be a sine or a cosine, depending on the value of $\vec{E}(t)$ at the origin at $t = 0$.) The magnetic field will have the exact same sine or cosine factor, with the same argument. The argument of this sine or cosine factor will have an ωt term and a kx, ky, or kz term. The spatial term tells you what the axis of propagation of the electromagnetic wave is (whether it flows in the x, y, or z direction). For the above field, the axis of propagation is the z axis. And in this case the electric and magnetic fields will be constant in each plane with $z =$ constant, which is why this is called a plane wave.

The *relative* sign of the temporal and spatial terms in the argument of the sine or cosine function tells us which direction the electromagnetic wave propagates. If the two terms have the same sign ($+kz$ and $+\omega t$ or $-kz$ and $-\omega t$) then the wave propagates in the negative direction along the axis. If the two terms have opposite signs, then the wave propagates in the positive direction. For the fields given above, the wave propagates in the $+\hat{z}$ direction. Think of the propagation direction as the direction you'd have to walk to stay on top of a particular peak as time moves forward ("surfing the electromagnetic wave"). So, in the above equation if I increase t, then to stay on top of the same peak I also have to increase z proportionately.

The final piece of the electric field is a unit vector which tells us which direction the field points (in this case the $\hat{x}$ direction). Note that if we have something which is *not* a unit vector in this slot (eg. $\hat{x} + \hat{y}$) then E_0 will not be the maximum length of the electric field vector. The magnetic field points in a direction perpendicular to the electric field. Given the direction of the electric field, we can find the magnetic field direction using the Poynting vector. (It was named after a guy called "Poynting", no joke!)

2.1.1 The Poynting Vector

The Poynting vector is defined as:

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} \tag{47}$$

It points in the propagation direction, so given the direction of the electric field and the direction of propagation, to find the direction of the magnetic field we simply compute the cross product:

$$\vec{B} = \hat{s} \times \frac{\vec{E}}{c} \tag{48}$$

where $\hat{s}$ is a unit vector in the propagation direction. Similarly, given the magnetic field and the propagation direction:

$$\vec{E} = c\vec{B} \times \hat{s} \tag{49}$$

The length of the Poynting vector is also an interesting quantity. It is the instantaneous intensity ($I(t)$, which is *not* a current) of the electromagnetic wave, which is the power carried in the wave per unit area (units are W/m^2):

$$I(t) = \left|\vec{S}(t)\right| = \frac{\left|\vec{E}(t)\right| \left|\vec{B}(t)\right|}{\mu_0} = \frac{\left|\vec{E}(t)\right|^2}{\mu_0 c} = \frac{c\left|\vec{B}(t)\right|^2}{\mu_0} \tag{50}$$

The quantity $Z_0 = \mu_0 c = 377\Omega$ is a constant of nature (the "impedance of free space", I guess) and has no particular relationship to any of the LRC circuit stuff.

Generally we are interested in the average intensity of the wave. Since the intensity is proportional to the square of the electric field, which varies sinusoidally, time-averaging introduces a factor of 1/2:

$$I = \langle I(t) \rangle = \frac{E_{\text{max}}^2}{2Z_0} = \frac{E_{\text{RMS}}^2}{Z_0} = \frac{cB_{\text{max}}^2}{2\mu_0} = \frac{cB_{\text{RMS}}^2}{\mu_0} = c\langle u \rangle \tag{51}$$

The last equality relates the average intensity of the electromagnetic field to the average energy density u in the electromagnetic field. This energy density (and its average value) can be computed using the equations for electric and magnetic energy density discussed in earlier exam reviews.

This intensity is, as aforementioned, a power per unit area. What this means is that a laser beam with cross-sectional area A_{beam} and an intensity I will have an average power:

$$P = A_{\text{beam}} I \tag{52}$$

That is, it will deposit energy on a piece of perfectly absorbent material in its path at a rate of P Joules per second.

If the area of the chunk of material in the beam path is greater than or equal to the area of the beam, then it will experience a force (due to the input of energy and momentum from the light beam) of $F_{\text{absorber}} = P/c$. If the material is smaller than the beam, it will only receive an amount of power proportional to its area, and so will experience a force:

$$F_{\text{absorber}} = \frac{I}{c} A_{\text{absorber}} \tag{53}$$

Note that if the material is perfectly reflective, then the light will bounce off with equal and opposite momentum to that with which it entered and so the force on the object will be twice as great:

$$F_{\text{reflector}} = 2\frac{I}{c} A_{\text{reflector}} \tag{54}$$

2.1.2 Linear Polarization

Linearly polarized light is exactly like the electric and magnetic fields described above. It has a constant direction, and its length oscillates sinusoidally. Its polarization direction is the direction of the *electric* field. Some examples of linearly polarized light are:

$$\vec{E}(t) = E_0 \sin(kz - \omega t)\hat{x} \tag{55}$$
$$\vec{B}(t) = B_0 \sin(kz - \omega t)\hat{y} \tag{56}$$

which is polarized in the $\hat{x}$ direction, and:

$$\vec{E}(t) = E_0 \sin(kz - \omega t)\left(\frac{\hat{x} + \hat{y}}{\sqrt{2}}\right) \tag{57}$$
$$\vec{B}(t) = B_0 \sin(kz - \omega t)\left(\frac{-\hat{x} + \hat{y}}{\sqrt{2}}\right) \tag{58}$$

which is polarized in the $(\hat{x} + \hat{y})/\sqrt{2}$ direction, ie. at $45°$ away from the x axis.

2.1.3 Circular Polarization

Circularly polarized light can be constructed by adding together two linearly polarized electromagnetic waves which are out of phase by 1/4 cycle and are polarized perpendicular to each other. An example of circularly polarized light is:

$$\vec{E}(t) = E_0(\cos(kz - \omega t)\hat{x} + \sin(kz - \omega t)\hat{y}) \tag{59}$$
$$\vec{B}(t) = B_0(-\sin(kz - \omega t)\hat{x} + \cos(kz - \omega t)\hat{y}) \tag{60}$$

Note that the electric and magnetic field vectors in this wave have constant lengths, but their directions rotate about the axis of propagation. Also note that each component ($\hat{x}$ and $\hat{y}$) carries, on average, half of the intensity of the EM wave.

There are two varieties of circularly polarized light. Right circularly polarized light rotates around its polarization direction according to a right-hand rule, left circularly polarized light rotates according to a left-hand rule. The above light is right circularly polarized

2.1.4 Unpolarized Light

Unpolarized light is made of a bunch of photons all with different polarizations. You can think of it as being fifty percent vertically polarized photons and fifty percent horizontally polarized photons. But remember that there's no particular relationship amongst the polarizations and phases (timing of the field peaks) of these photons.

2.2 Polarizers (AKA Polaroids)

A polarizer is a sheet of material which allows the component of the electric field which points along its transmission axis to pass through undisturbed, while the component which is perpendicular to its transmission axis is completely blocked. All the light which comes out of a polarizer will be linearly polarized along the transmission axis, regardless of what the input polarization is.

For unpolarized and circularly polarized light, half the light intensity is in light polarized along the TA and the other half is in light polarized perpendicular to the TA. So if I_0 is the original intensity of the light, the intensity I_1 after it passes through the polarizer will be:

$$I_1 = \frac{1}{2} I_0 \tag{61}$$

For linearly polarized light, the output light intensity will be:

$$I_1 = I_0 \cos^2(\Delta\theta) \tag{62}$$

where $\Delta\theta$ is the angle between the polarization direction of the light (its electric field direction) and the transmission axis of the polaroid. Note that the sign of $\Delta\theta$ is irrelevant, since $\cos(\theta) = \cos(-\theta)$.

2.3 Wave Plates

A wave plate is made of a material which has one index of refraction for light polarized along one axis and a different index of refraction for light polarized along the perpendicular axis. (This kind of material is called birefringent.) Because of these two different indices of refraction, the components of the light in each direction will travel through the plate at different speeds, creating a phase difference between the two components of the light. The so-called fast axis has a smaller index of refraction than the slow axis ($n_f < n_s$) so that the light polarized along the fast axis travels faster. In particular, if this plate has thickness d, the slow-axis light will fall behind the fast-axis light by a time of

$$\Delta t = \frac{d}{v_s} - \frac{d}{v_f} = \frac{d}{c/n_s} - \frac{d}{c/n_f} = \frac{d}{c}\left(n_s - n_f\right) \tag{63}$$

For example, a quarter-wave plate is designed to shift the slow-axis light phase backwards by $\pi/2 = 90°$ with respect to the fast-axis light for light of some particular frequency f. Note that a QWP is only a QWP for light at this frequency, since only at the frequency f will Δt, which is an intrinsic property of the plate, be exactly equal to 1/4 of the period of oscillation of the light ($T = 1/f$). If we double the frequency to $f' = 2f$, then it will instead act as a half-wave plate, since

$$\Delta t = \frac{1}{4}T = \frac{1}{4}\left(\frac{1}{f}\right) = \frac{1}{4}\left(\frac{1}{f'/2}\right) = \frac{1}{2f'} = \frac{1}{2}T' \tag{64}$$

If unpolarized light is put through a QWP it will remain unpolarized, since there is no specific phase relation between the components. If we put circularly polarized light through, it will become linearly polarized at an angle of 45° from the axes, and conversely, if we put through light which is linearly polarized at 45° to the axes, it will come out circularly polarized. Light which is linearly polarized along either the fast or slow axis will be unaffected by a quarter-wave plate, since the relative phase shift requires components along both axes. Finally, light which is linearly polarized at some other angle to the axes will be partly circularly polarized and partly linearly polarized. This is called elliptically polarized light.

A half-wave plate consists of two identical quarter-wave plates glued together. Its effect is to shift the slow-axis phase backwards by $\pi = 180°$. Effectively, this reverses the sign of the slow-axis component with respect to the sign of the fast-axis component. Unpolarized light, of course, will still be unaffected. Circularly polarized light will have the direction of its rotation reversed (LCP$\rightarrow$RCP and *vice versa*). Light linearly polarized at an angle of θ with respect to the fast axis will have its polarization angle converted to $-\theta$ (eg. it will go from 36° above the x axis to 36° below it).

A full-wave plate is two identical half-wave plates stuck together. It shifts the slow-axis phase backwards by $2\pi = 360°$ and has no net effect.

A wave plate will not (by itself) change the intensity of light passing through it.

2.4 Trivia

A solution of biologically-formed sugar (or some other molecules) is also birefringent, but instead of changing the relative phases of the two linear polarizations it changes the relative phase of the two circular polarizations.

What this means is that circularly polarized light passing through a sugar solution will be unaffected, while linearly polarized light will remain linearly polarized, but its polarization will be rotated. The amount by which the polarization is rotated will depend on the amount of dissolved sugar per unit volume as well as on the distance the light travels through the solution.

Unpolarized light will, of course, be unaffected.

2.5 Doppler Shifts

If the relative speed of an observer and a light source is very high, the freqency (and wavelength) of the light the observer sees will appear to change (since the speed of the light must be a constant value c).

If the observer and the light source are moving towards each other, the collision between them is in some sense "more energetic" than it would be if they were stationary. So the light the observer sees is more excited: it has higher energy, oscillates faster (higher frequency), and has a shorter wavelength. This is called a blueshift, since blue light is at the high-energy, short-wavelength end of the visible spectrum. (Of course, in a vacuum, the observed light will *always* have a speed c.)

Conversely, if the observer and light source are moving apart, the light has to expend a little of its energy to "catch up with" the observer. So it has lower energy when the observer sees it: it oscillates slower and has a longer wavelength. This is called a redshift.

If the relative speed is not comparable to the speed of light, we can use the following approximation formula to calculate the observed frequency and wavelength:

$$\frac{|\lambda_{\text{obs}} - \lambda_{\text{em}}|}{\lambda_{\text{em}}} = \frac{|f_{\text{obs}} - f_{\text{em}}|}{f_{\text{em}}} \approx \frac{v}{c} \tag{65}$$

where the sign of the change in the wavelength or frequency can be determined by the criteria above.

If the relative speed between the observer *is* comparable to the speed of light, then we must use the exact formula:

$$\frac{|\lambda_{\rm obs}^2 - \lambda_{\rm em}^2|}{\lambda_{\rm obs}^2 + \lambda_{\rm em}^2} = \frac{|f_{\rm obs}^2 - f_{\rm em}^2|}{f_{\rm obs}^2 + f_{\rm em}^2} = \frac{v}{c} \tag{66}$$

where again, the sign of the changes in wavelength and frequency are determined by physical considerations.

2.6 More Trivia

Light at different wavelengths is often referred to by different names, but of course all of these wavelengths are still *light* even though they may not be visible light. Under many circumstances, these different kinds of light have qualitatively different interactions with matter, which make the distinctions useful, although the boundaries between the different kinds of light are only approximate. The different types of light are:

region	λ	f (Hz)	energy per photon (eV)
radio waves	> 1 cm	$< 3 \cdot 10^9$	$< 10^{-5}$
microwaves	100 μm – 1 cm	$3 \cdot 10^9$ – $3 \cdot 10^{12}$	10^{-5} – 0.01
infrared light	780 nm – 100 μm	$3 \cdot 10^{12}$ – $3.8 \cdot 10^{14}$	0.01 – 1.6
visible light	390 nm – 780 nm	$3.8 \cdot 10^{14}$ – $7.7 \cdot 10^{14}$	1.6 – 3.3
ultraviolet light (UV)	1 nm – 390 nm	$7.7 \cdot 10^{14}$ – $3 \cdot 10^{17}$	3.3 – 10^3
X-rays	0.1 Å– 1 nm	$3 \cdot 10^{17}$ – $3 \cdot 10^{19}$	10^3 – 10^5
gamma rays	< 0.1 Å	$> 3 \cdot 10^{19}$	$> 10^5$

Note that 1 m $= 10^2$ cm $= 10^3$ mm $= 10^6$ μm $= 10^9$ nm $= 10^{10}$ Å.

The approximate color ranges for visible light are:

color	λ (nm)	f (THz)	energy per photon (eV)
red	622 – 780	384 – 482	1.59 – 1.99
orange	597 – 622	482 – 503	1.99 – 2.08
yellow	577 – 597	503 – 520	2.08 – 2.15
green	492 – 577	520 – 610	2.15 – 2.52
blue	455 – 492	610 – 659	2.52 – 2.73
violet	390 – 455	659 – 769	2.73 – 3.18

The units of frequency here are terahertz, 1 THz $= 10^{12}$ Hz.

3 Reflection and Refraction

3.1 Angles

In all cases, the angles of interest in reflection and refraction problems are the angle between the light beam and the *perpendicular* to the surface it hits. The angle between the light beam and the surface itself is not at all useful and should be immediately be converted into its complement. (The complement of an angle θ is $90^\circ - \theta$.)

3.2 Reflection

Whenever light is reflected, angle of incidence (θ_i) always equals angle of reflection (θ_rl). No exceptions, no caveats, no nuthin'.

3.3 Refraction

When light travelling in one medium (eg. air) is incident on the surface of another, transparent medium (eg. water), two things will happen. A small portion of the light will always be reflected with $\theta_\mathrm{i} = \theta_\mathrm{rl}$. Usually, the rest of the light (most of it) will be transmitted into the medium, but it will bend away from its original path. The angle of the transmitted light beam, with respect to the normal to the surface, can be determined by using the rule that light always takes the quickest path between two points. As discussed above, light in a medium with index of refraction n will have a speed c/n, so in the two different media it will travel at different speeds, and the quickest path between a point in one medium and a point in the other will *not* be a straight line, but will instead have the light travel a slightly longer distance in the lower-n medium in exchange for a shorter distance in the higher-n medium. Applying this rule and working out the geometry will give Snell's law:

$$n_\mathrm{i} \sin\theta_\mathrm{i} = n_\mathrm{rr} \sin\theta_\mathrm{rr} \tag{67}$$

where n_i and n_rr are the indices of refraction of the two media and θ_i and θ_rr are the angles of the light paths in the two media.

If the medium in which the light starts has a lower index of refraction than the medium into which it is travelling, then the angle of refraction will be less than the angle of incidence. Conversely, if the light starts in the higher-n medium, its angle of refraction will be larger than the angle of incidence. This can cause a breakdown in Snell's law, because in this case, there will be some angle of incidence, $\theta_\mathrm{c} < 90^\circ$ for which the angle of refraction is 90°. If we try to increase the incident angle beyond this angle, all of the light will be totally internally reflected in the higher-n medium, with $\theta_\mathrm{i} = \theta_\mathrm{rl}$. The angle of incidence at which this switch between refraction and total internal reflection occurs is called the critical angle, and by setting the angle of refraction to 90° in Snell's law we can compute that the critical angle is:

$$\theta_\mathrm{c} = \sin^{-1}\left(\frac{n_\mathrm{rr}}{n_\mathrm{i}}\right) \tag{68}$$

Note that if $n_{\rm rr} > n_{\rm i}$ then in computing $\theta_{\rm c}$ we are attempting to compute the inverse sine of something larger than one, which is impossible. So we can never have total internal reflection if the light starts in the lower-n medium.

3.4 Trivia

Light reflected from an interface between two media will always be polarized parallel to the surface. So, for example, if you are staring at a lake and sunlight is being reflected off the lake, the reflected light will be horizontally polarized. If you want to block the reflected light, you need to use a polarizer with its transmission axis oriented vertically.