Physics 212 Exam 1 Review Notes

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1 Electrostatics and Gauss' Law

1.1 Constants

$$k = 1/4\pi\varepsilon_0 = 9 \cdot 10^9\ \mathrm{Nm^2/C^2} = \text{Coulomb constant}$$

$$\varepsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C^2/Nm^2} = \text{permittivity of free space}$$

$$e = 1.60 \cdot 10^{-19}\ \mathrm{C} = \text{magnitude of the charge on an electron}$$

1.2 Basics

$$\vec{F} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{21}^2}\hat{r} = \frac{k q_1 q_2}{r_{21}^2}\hat{r} \tag{1}$$

This is the electrical force between charges q_1 and q_2 which are separated by distance r_{21}. The force is directed along the line between the charges. The units of force are Newtons, $1\ \mathrm{N} = 1\ \mathrm{kg \cdot m/s^2}$.

$$\vec{E} = \frac{kq}{r^2}\hat{r} \tag{2}$$

This is the electrical field due to the charge q at a point P located a distance r from the charge. It is directed along the line between the charge and the point. If a second charge q_0 was placed at the point P, then the charge q_0 would experience a force $\vec{F} = q_0\vec{E}$. The units of electric field are $\mathrm{N/C} = \mathrm{V/m}$.

$\vec{E} \equiv 0$ in the bulk of a conductor (*ie.* a metal), although not necessarily in an air pocket inside the conductor.

Note that the x, y, and z components of the electric force on q_1 are:

$$\begin{aligned}
\vec{F}_x &= \frac{kq_1q_2}{r_{21}^2}\left(\frac{r_x}{r_{21}}\right)\hat{x} \\
\vec{F}_y &= \frac{kq_1q_2}{r_{21}^2}\left(\frac{r_y}{r_{21}}\right)\hat{y} \\
\vec{F}_z &= \frac{kq_1q_2}{r_{21}^2}\left(\frac{r_z}{r_{21}}\right)\hat{z}
\end{aligned}$$

where r_x, r_y, and r_z are the x, y, and z components of the vector $\vec{r}_{21}$ pointing from q_2 to q_1. The electric field components at the position of q_1 have a similar relationship.

If we have several point charges (or several charged insulating objects) and we want to find the net electric field due to these objects at a particular point P, then we use the superpostion principle.

Each individual object, in isolation, would create a field $\vec{E}_i(\vec{r}_{i\to P})$, (where $\vec{r}_{i\to P}$ is the distance vector from the object to the point P) and the total field is just the sum of all these individual fields:

$$\vec{E}(P) = \sum_i \vec{E}(\vec{r}_{i\to P}) \tag{3}$$

The components of the total field are the sum of the components of all the individual fields:

$$\vec{E}_x(P) = \sum_i \vec{E}_x(\vec{r}_{i\to P}) \tag{4}$$

$$\vec{E}_y(P) = \sum_i \vec{E}_y(\vec{r}_{i\to P}) \tag{5}$$

$$\vec{E}_z(P) = \sum_i \vec{E}_z(\vec{r}_{i\to P}) \tag{6}$$

1.3 Gauss' Law

$$\Phi_E \equiv \int_A \vec{E}\cdot d\vec{A} \tag{7}$$

This is the definition of the electric flux through the surface A, where $d\vec{A}$ is a vector whose length is an infinitesimal area surrounding the point on the surface where we are measuring the electric field and whose direction is the outward perpendicular to the surface at that point. $\Phi_E > 0$ if the net flux through the surface is outward and it's negative if the net flux is inward. The units of electric flux are Nm^2/C.

$$\Phi_E = \frac{Q_{\mathrm{enc}}}{\varepsilon_0} \tag{8}$$

This is Gauss' law. Q_{enc} is the *total* charge enclosed by the surface A. Note that if there are two point charges inside the surface, one with $+Q$ Coulombs and the other with $-Q$, then the total enclosed charge is zero, so the net flux through the surface is zero. Similarly, if the only charge in the universe is a point charge of $+Q$, but this charge is *not* enclosed by the surface, then there is no total enclosed charge and so no net flux. However, different pieces of the surface may still have non-zero flux. For example, if you have a cylinder, the curved surface and one end-cap may have positive flux, while the other end-cap has flux equal to the negative of the total flux through the other two surfaces: $\Phi_{\mathrm{end2}} = -(\Phi_{\mathrm{end1}} + \Phi_{\mathrm{round}})$. But if you add up the fluxes through all the parts of the surface, the total will be zero.

Gauss' law allows the computation of the electric field in certain symmetrical situations. In each case, the procedure begins with drawing a Gaussian surface which passes through the point where you want to find the electric field and which shares the symmetry of the charge distribution. Then we compute the flux through this surface in two different ways: using the definition of flux, and using Gauss' law. If we have chosen our Gaussian surface well, the electric field will have constant magnitude on and be perpendicular to certain parts of the surface (the parts where we want to know the electric field, which have total area A_{care}), and will be zero on or tangent to the rest of the surface (the parts where we don't care about the field, which have total area A_{rest}). Then the definition of flux tells us that $\Phi_E = E_{\mathrm{care}}A_{\mathrm{care}}$, since the surface has been designed so that the flux through the other regions is zero. Next we count the total enclosed charge inside the surface, Q_{enc}, which, using Gauss' law, gives: $\Phi_E = Q_{\text{enc}}/\varepsilon_0$. We can then equate these two formulae for the flux and solve for the electric field at the point in question:

$$E_{\text{care}} = \frac{Q_{\text{enc}}}{\varepsilon_0 A_{\text{care}}} \tag{9}$$

Below we discuss the computation of Q_{enc} and A_{care} for the various possible symmetries.

1.3.1 Spherical Symmetry

For a spherically symmetric charge distribution, the electric field points radially outward or radially inward at each point and its value at a particular point depends only on the distance of that point from the center of the distribution. So to compute the field at a radius r, the appropriate Gaussian surface is a sphere of radius r concentric with the distribution. It will have area:

$$A = 4\pi r^2 \tag{10}$$

which is equal to the area A_{care} used in the field computation, and the enclosed charge will depend on the radius from the center. Some common spherical objects are described below. Note that if the electric field at any radius is zero then the *net* charge inside that radius is zero.

For a point charge Q, the enclosed charge at any radius is simply:

$$Q_{\text{enc}} = Q \tag{11}$$

The total charge in a solid insulating sphere of radius a and volume charge density ρ (C/m^3) is:

$$Q = \frac{4}{3}\pi a^3 \rho \tag{12}$$

So if we want the electric field at a distance r from the center of such an object then we use:

$$Q_{\text{enc}} = \frac{4}{3}\pi r^3 \rho = \left(\frac{r}{a}\right)^3 Q \text{ for } r < a \tag{13}$$

$$Q_{\text{enc}} = \frac{4}{3}\pi a^3 \rho = Q \text{ for } r > a \tag{14}$$

For a thin hollow spherical insulating shell of radius a with an area (or "surface") charge density σ (C/m^2) (or for the inner or outer surface of a metal sphere) the total charge is:

$$Q = 4\pi a^2 \sigma \tag{15}$$

The enclosed charge at a radius r from the center is:

$$Q_{\text{enc}} = 0 \text{ for } r < a \tag{16}$$

$$Q_{\text{enc}} = 4\pi a^2 \sigma = Q \text{ for } r > a \tag{17}$$

For a hollowed insulating sphere with inner radius a, outer radius b, and volume charge density ρ (C/m^3) the total charge is:

$$Q = \frac{4}{3}\pi (b^3 - a^3)\rho \tag{18}$$

The enclosed charge at a radius r from the center is:

$$Q_{\text{enc}} = 0 \text{ for } r < a \tag{19}$$

$$Q_{\text{enc}} = \frac{4}{3}\pi\left(r^3 - a^3\right)\rho = \left(\frac{r^3 - a^3}{b^3 - a^3}\right) Q \text{ for } a < r < b \tag{20}$$

$$Q_{\text{enc}} = \frac{4}{3}\pi\left(b^3 - a^3\right)\rho = Q \text{ for } r > b \tag{21}$$

For any spherically symmetric charge distribution, Gauss' law tells us that the electric field at a radius r from the center is simply:

$$\vec{E} = \frac{kQ_{\text{enc}}}{r^2}\hat{r} \tag{22}$$

where Q_{enc} is the total charge within a radius r of the center.

Note that if we have a metal sphere of radius b with an air pocket in the center of radius a, then the electric field in the actual metal will be zero, so in order to ensure this, the surface charge on the inner surface of the sphere will be equal to the negative of the total charge q trapped inside the air pocket. (Then Gauss' law says that $E = kQ_{\text{enc}}/r^2 = k(q-q)/r^2 = 0$ for $a < r < b$.) Whatever charge is leftover goes to the outer surface. So if the sphere has a net charge Q, then the outer surface will have a total charge $Q + q$. So, if the trapped charge is a a point charge q at the center of the sphere then the electric field at a radius r will be:

$$\vec{E} = \frac{kq}{r^2}\hat{r} \text{ for } r < a \tag{23}$$

$$\vec{E} = 0 \text{ for } a < r < b \tag{24}$$

$$\vec{E} = \frac{k(Q+q)}{r^2}\hat{r} \text{ for } r > b \tag{25}$$

1.3.2 Cylindrical Symmetry

Again, the electric field points radially outward or radially inward at each point, and its value depends only on the distance from the axis of the charge distribution. So the appropriate Gaussian surface to find the field at radius r is a cylinder of radius r concentric with the charge distribution and whose length is equal to the length ℓ of the charge distribution. For an infinite object, we pretend it has a finite but unknown length ℓ in order to make computations, and this ℓ will generally cancel out in the end. The cylinder has a curved surface and two end-caps, but the end-caps will be tangent to the electric field and so do not contribute to the flux. So the area used for computation of the electric field is:

$$A_{\text{care}} = 2\pi r\ell \tag{26}$$

The total charge in such a cylindrically symmetric object is:

$$Q = \lambda\ell \tag{27}$$

where λ (C/m) is the charge per unit length on the object. (For example, if a one-centimeter-long segment of the charged object has a charge of 1 Coulomb, then the charge per unit length is

$\lambda = 1$ C/cm $= 100$ C/m.) So the interesting parameter for finding the electric field at a particular radius is actually the charge per unit length which is inside that radius.

In particular, a thin charged wire will simply be described as having a charge per unit length λ, so that

$$\lambda_{\rm enc} = \lambda \text{ for all } r \tag{28}$$

For a solid insulating cylinder with radius a and volume charge density ρ (C/m^3) the charge per unit length is:

$$\lambda = \pi a^2 \rho \tag{29}$$

and the charge inside a radius r is:

$$\lambda_{\rm enc} = \pi r^2 \rho = \left(\frac{r}{a}\right)^2 \lambda \text{ for } r < a \tag{30}$$

$$\lambda_{\rm enc} = \pi a^2 \rho = \lambda \text{ for } r > a \tag{31}$$

A thin cylindrical insulating shell of radius a (or the inner or outer surface of a cylindrical piece of metal) with area ("surface") charge density σ (C/m^2) has a charge per unit length:

$$\lambda = 2\pi a \sigma \tag{32}$$

so that the charge inside a radius r is:

$$\lambda_{\rm enc} = 0 \text{ for } r < a \tag{33}$$

$$\lambda_{\rm enc} = 2\pi a \sigma = \lambda \text{ for } r > a \tag{34}$$

For a solid insulating cylinder with its center cut out, inner radius a, outer radius b, and volume charge density ρ (C/m^3) the total charge per unit length is:

$$\lambda = \pi(b^2 - a^2)\rho \tag{35}$$

while the charge inside a radius r is:

$$\lambda_{\rm enc} = 0 \text{ for } r < a \tag{36}$$

$$\lambda_{\rm enc} = \pi(r^2 - a^2)\rho = \left(\frac{r^2 - a^2}{b^2 - a^2}\right) \lambda \text{ for } a < r < b \tag{37}$$

$$\lambda_{\rm enc} = \pi(b^2 - a^2)\rho = \lambda \text{ for } r > b \tag{38}$$

For such a charge distribution, Gauss' law gives the electric field at a distance r from the axis to be:

$$\vec{E} = \frac{\lambda_{\rm enc}}{2\pi\varepsilon_0 r}\hat{r} \tag{39}$$

where $\lambda_{\rm enc}$ is the total charge per unit length within a distance r of the axis.

For a solid metal cylinder with charges trapped in an air pocket in the center, the charges on its inner and outer surfaces are similar to those for the spherical case.

1.3.3 Planar Symmetry

For a single infinite uniformly charged plane, the electric field points perpendicular to the plane, outward if it is positively charged and inward if it is negatively charged. The appropriate Gaussian surface is a "pillbox" or film cannister shaped object, basically a cylinder whose curved surface is perpendicular to the plane and whose endcaps (of area A) are parallel to it. Since the curved surface is parallel to the electric field, it does not figure into the computation of electric flux and the total relevant area for the flux computation is:

$$A_{\text{care}} = 2A \tag{40}$$

The enclosed charge inside this Gaussian pillbox will be:

$$Q_{\text{enc}} = A\sigma \tag{41}$$

where σ is the charge per unit area on the plane and is measured in $\mathrm{C/m^2}$. So the charge per unit area is the interesting quantity for Gauss' law computations.

For a thin insulating sheet or a single surface of a metal slab, the charge per unit area is simply given as the surface charge density σ.

The total charge per unit area on a metal slab is the sum of the surface charges on both sides of it:

$$\sigma = \sigma_L + \sigma_R \tag{42}$$

For a solid insulating slab of thickness d with charge per unit volume ρ, the charge per unit area is:

$$\sigma = \rho d \tag{43}$$

So from Gauss' law, the electric field anywhere outside a single uniformly charged sheet or slab oriented in the yz plane is:

$$\vec{E} = \pm\frac{\sigma}{2\varepsilon_0}\hat{x} \tag{44}$$

where the field points away from the slab if $\sigma > 0$ and towards the slab if $\sigma < 0$.

For a collection of charged sheets or slabs, the electric field is:

$$\vec{E} = \frac{\sigma_\ell - \sigma_r}{2\varepsilon_0}\hat{x} \tag{45}$$

where σ_ℓ is the (signed) sum of the surface charge densities of *all* objects to the left of the point of interest (eg. $\sigma_\ell = \sigma_{\ell 1} + \sigma_{\ell 2} = 3\ \mathrm{C/m^2} + (-2\ \mathrm{C/m^2}) = 1\ \mathrm{C/m^2}$), and σ_r is the sum of the surface charge densities of *all* objects to the right of the point of interest.

If $\sigma_\ell - \sigma_r > 0$ then the field points to the right, and if $\sigma_\ell - \sigma_r < 0$ then the field points left. (eg. $\sigma_\ell = 1\ \mathrm{C/m^2}$, $\sigma_r = -3\ \mathrm{C/m^2} \Rightarrow \sigma_\ell - \sigma_r = 4\ \mathrm{C/m^2} > 0$)

Note that the field inside a metal slab is always 0, and we can use this fact to find the charges on its two surfaces, σ_L and σ_R. Since we have two unknown quantities, we need two equations in order to solve for them. First of all, we are usually given that the total charge per unit area on the slab is σ_m, and this must be equal to the sum of the two surface charges:

$$\sigma_m = \sigma_L + \sigma_R \tag{46}$$

Next, since the electric field inside the slab is zero, we can write:

$$\frac{(\sigma_\ell + \sigma_L) - (\sigma_r + \sigma_R)}{2\varepsilon_0} = 0 \tag{47}$$

where σ_ℓ and σ_r are the total surface charges on all the external objects to the left and right of the slab, respectively. Solving these two equations gives:

$$\sigma_L = \frac{1}{2}(\sigma_m - \sigma_\ell + \sigma_r) \tag{48}$$

$$\sigma_R = \frac{1}{2}(\sigma_m + \sigma_\ell - \sigma_r) \tag{49}$$

Note that if the total charge density on the metal is $\sigma_m = 0$ then $\sigma_L = -\sigma_R$, and that if we add a net charge to the metal it is then distributed evenly between the two surfaces.

1.4 Electric Potential

Unlike electric force and electric fields, the electric potential is a scalar ("just a number"), so don't try to break it down into components or worry about directions when you're adding potentials. Just add them. The only way two potentials can cancel each other out is if they have opposite signs, for example, if they are generated by opposite-signed charges with the same magnitude and located at the same distance from the point in question. The units of electric potential are Volts, $1\ \text{V} = 1\ \text{Nm/C}$.

The following equation shows how to compute the amount by which the potential increases when you go from point A to point B. It's equal to the work done when you carry a charge q_0 from point A to point B, divided by q_0, and it's also equal to the negative integral of the electric field along *any path* from A to B (since the electric force is conservative).

$$V_B - V_A = \frac{W_{AB}}{q_0} = -\int_A^B \vec{E} \cdot d\vec{\ell} \tag{50}$$

Conversely, the electric field associated with a given electric potential V is:

$$\vec{E} = -\vec{\nabla} V = -\left(\frac{dV}{dx}\hat{x} + \frac{dV}{dy}\hat{y} + \frac{dV}{dz}\hat{z}\right) \tag{51}$$

1.4.1 Special cases

If $\vec{E}$ is constant and points from A to B and d is the distance from A to B:

$$V_B - V_A = -Ed \tag{52}$$

If the source of the potential is a point charge q at the origin:

$$V_B - V_A = \frac{q}{4\pi\varepsilon_0 r_B} - \frac{q}{4\pi\varepsilon_0 r_A} \tag{53}$$

If the source of the potential is a thin, infinitely-long wire with charge per unit length λ:

$$V_B - V_A = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r_A}{r_B}\right) \tag{54}$$

If A and B are both points within the same chunk of conductor:

$$V_B - V_A = 0 \tag{55}$$

1.4.2 "Absolute" Potential

Be aware that the electric potential V is only physically meaningful in terms of potential *differences* between two points. If you are asked for "the potential at point P", then you should have been given "the potential" at some reference point R. Use the equations above to find the potential difference between P and R, and then set $V_P = (V_P - V_R) + V_R$.

Also note that the electric potential is *continuous* everywhere (its value doesn't have sudden jumps), and the potential is *constant* (although not necessarily zero) everywhere inside a conductor. (Don't forget that the wires in an electrical circuit are conductors!)

For a single point charge located at the origin:

$$V(\vec{r}) = \frac{kq}{r} + V(\infty) \tag{56}$$

where $V(\infty)$ is the electric potential at infinity.

For a collection of point charges $\{q_i\}$ at distances $\{r_i\}$ from the point where the potential is desired:

$$V(\vec{r}) = V(\infty) + \sum_i \frac{kq_i}{r_i} \tag{57}$$

where $V(\infty)$ is the potential at infinity.

1.4.3 The Sign of the Potential or Potential Difference

Potential is sort of a measure of the "happiness" of a positive charge placed at that point. Positive charges like to fall down potential hills, so they're happier at lower potential. Further, we know that electric fields push positive charges in the direction of the field lines. So electric fields point from higher potential to lower potential. And current (which is made of positive chargese) will flow from higher potential to lower potential.

So if you need to find the sign of a potential difference $V_B - V_A$, then check the above conditions. If $V_B > V_A$ the potential difference will be positive. If you need to find the sign of an absolute potential V, then you need to look at the reference point where $V_R = 0$. If a positive charge is happier at the reference point then $V > 0$; if a positive charge is less happy at the reference point then $V < 0$.

(Note that negative charges are actually happier at *higher* potentials.)

1.4.4 Computing Potentials or Potential Differences in Complex Situations

You may be asked to compute absolute potentials or potential differences between two points in a complicated arrangement of charges or in an electrical circuit. Since the electric potential is defined as a *path* integral, you first need to choose the starting and ending points of your path. If you are asked to find the potential difference $V_B - V_A$ between points B and A, then the starting point will be A while the ending point will be B. (Later you will pretend that the "absolute" potential at point A is $V_A = 0$ to make things easier to keep track of.) If you are asked to find the "absolute" potential at a point P with the potential at R given as V_R, then R is the start point and P is the end point. (Note that for an electrical circuit, the ground of the circuit, or, if there's only one battery, the negative terminal of the battery, are usually given as the reference point and are considered to be at zero potential.)

The next step is to choose a path between the two points. Since the electric force is conservative, in principle any path between them whatsoever will do, but in practice some paths are easier to compute than others. For circuits, the path should of course be along one of the wire tracks. In this case the electric potential will be constant along the wires and will increase or decrease only when crossing from one side to the other of a circuit element. For other arrangements a path composed of segments parallel to or perpendicular to the electric field is generally the simplest route to take. In this case, the potential will be constant on the perpendicular segments and will vary continuously along the parallel segments. Note also that the potential may vary differently in different regions, depending on what the electric field is in each region. (For example, it will be constant inside a conductor, since the electric field is zero there.)

Finally, you are ready to integrate along the path. Starting at point A or the reference point, write down the potential there (using the assumptions discussed above). Now look at the first segment of the path. *Integrate* the electric field along this path segment, according to the defining equation (50) for electrical potential. (Note that if the electric field varies at all along the segment then it is *not* appropriate to just multiply the electric field at some point on the segment by the segment's length. This will almost always give you the wrong answer.) For electric circuit elements, the magnitude of the potential difference will just be the potential V across the circuit element, for example $V = IR$ for a battery.

Then determine if the potential increases or decreases along this path segment (moving from A or R towards B or P, respectively), using the sign criteria above. Don't forget that an electrical current is a flow of positive charges, so the direction of current flow in a circuit tells you about the sign of potential differences. If the potential increases along the path segment, then you should add the potential change to the starting potential. If it decreases, then you should subtract the change from the starting potential.

Repeat this process along each segment until you reach the endpoint (B or P) of the path and you'll have your answer.

1.5 Potential Energy

The following equation gives the electric potential energy of a charge q_0 at a point P where the electric potential is V_P. Energy is, of course, measured in Joules, 1 J = 1 Nm.

$$U = q_0 V_P \tag{58}$$

Note that, like gravitational potential energy, electric potential energy is only meaningful with respect to the potential energy at some reference point. For example, on earth we might choose to say that the potential energy of an object is zero when it is located at earth's surface. The potential energy will then increase as we elevate the object above the surface of the earth. So an object of mass m raised to a height h has gravitational potential energy $U_g = mgh$, and if we allowed it to fall to the ground it would gain kinetic energy $T = U_g$ as it fell. Similarly, the charge q_0 in the above equation has an electric potential energy $q_0 V_P$ with respect to a reference point R where the electric potential has been chosen to be $V_R = 0$. So if we released this charge, it would fall towards R and would gain a kinetic energy $T = q_0(V_P - V_R) = q_0 V_P$ in the process. Of course, if $q_0 V_P < 0$ then this means that the charge would already have to have kinetic energy $T = |q_0 V_P|$ when it started at point P in order to be able to reach point R and that it would lose all of this kinetic energy *en route*.

Another way to think of electric potential is to imagine that we have one completely isolated charge, q_1, in a region of no electric field. If we release this charge, it will not go anywhere at all, so we can think of it as having no electric potential energy. Now we bring in a second charge, q_2 and place it at a distance r_{12} from q_1. If we hold q_2 fixed and release q_1, q_1 will either fly off to infinity or fall towards q_1, depending on their relative signs. The kinetic energy it gains as it flies off to infinity, or the kinetic energy it could have gained by falling from infinity to r_{12}, is due to a change in its electric potential energy with respect to the situation where it was isolated. So we can write that the electric potential energy stored in the arrangement of a pair of charges, q_1 and q_2, separated by a distance r_{12} is:

$$U_{12} = \frac{k q_1 q_2}{r_{12}} \tag{59}$$

In general, if we have two or more charges, then the potential energy is:

$$U_n = \frac{1}{2} \sum_{i,j} \frac{k q_i q_j}{r_{ij}} \tag{60}$$

where the sum is over all possible pairs (($i = 1, j = 2$) and ($i = 2, j = 1$) are both included).

The change in potential energy when a charge is moved from point A to point B is equal to the work done in moving the charge between those two points. If the potential energy *increases* then you must do *positive* work on the charge to push it up the potential hill. If the potential energy *decreases*, then one would have to do *negative* work on the charge. In other words, you can actually use the charge's motion down the potential hill to do work on another object.

2 Capacitors and (DC) RC Circuits

2.1 Basics

$$V = \frac{Q}{C} \tag{61}$$

This is the defining equation relating the voltage across a capacitor to the charge on the capacitor and its capacitance. The units of capacitance are Farads, 1 F = 1 C/V = 1 C^2/Nm.

The total stored energy in a capacitor when it's charged up to a charge Q is:

$$U = \frac{1}{2}CV^2 = \frac{Q^2}{2C} = \frac{1}{2}QV \tag{62}$$

The power dissipated in a capacitor is:

$$P_C \equiv 0 \tag{63}$$

(Capacitors *never* dissipate power.) The units of power are Watts, 1 W = 1 J/s = 1 Nm/s.

2.2 Series and Parallel

The total capacitance of two capacitors in parallel is:

$$C = C_1 + C_2 \tag{64}$$

Note that $C > C_1$, $C > C_2C > C_1$, $C > C_2$, and that the charge on the equivalent capacitor is $Q = Q_1 + Q_2$. The voltage across the equivalent capacitor is $V = V_1 = V_2$.

The total capacitance of two capacitors in series is:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \tag{65}$$

Note that $C < C_1$, $C < C_2$, and that the charge on the equivalent capacitor is $Q = Q_1 = Q_2$. The voltage across the equivalent capacitor is $V = V_1 + V_2$.

2.3 Constructing a Capacitor

The capacitance of a parallel plate capacitor of area A and separation d which is filled with air or vacuum is:

$$C = \frac{\varepsilon_0 A}{d} \tag{66}$$

The following gives the capacitance of a concentric spherical capacitor filled with vacuum or air, where the radius of the inner spherical conductor is a while the outer spherical conductor has radius b.

$$C = \frac{4\pi\varepsilon_0 ab}{(b-a)} \tag{67}$$

You may be asked to find the capacitance of a single sphere. Then you just take the limit of this formula as $b \to \infty$, giving $C = 4\pi\varepsilon_0 a$.

The capacitance of a concentric cylindrical capacitor of length ℓ filled with vacuum or air, where the inner conductor has radius a and the outer conductor has radius b, is:

$$C = \frac{2\pi\varepsilon_0 \ell}{\ln(b/a)} \tag{68}$$

You may be asked to find the capacitance per unit length of an infinite cylindrical conductor. Then you just divide both sides of this formula by ℓ, giving $C/\ell = 2\pi\varepsilon_0 / \ln(b/a)$.

$$C_\kappa = \kappa C \tag{69}$$

If we take a capacitor of capacitance C which is filled with air or vacuum and insert a dielectric of capacitance κ then this formula gives the new capacitance:

$$C_\kappa = \kappa C \tag{70}$$

Note that if you have a parallel plate capacitor with plate separation d and area A and you insert a dielectric with thickness $d/2$ and area A then this capacitor can be treated as two capacitors in series:

$$\frac{1}{C'} = \frac{1}{C_1} + \frac{1}{C_2} = \left(\frac{\varepsilon_0 A}{d/2}\right)^{-1} + \left(\kappa\frac{\varepsilon_0 A}{d/2}\right)^{-1} = \left(\frac{\varepsilon_0 A}{d}\right)^{-1}\left(\frac{1}{2} + \frac{1}{2\kappa}\right) = \frac{1}{C}\cdot\frac{1}{2}\left(\frac{1+\kappa}{\kappa}\right)$$

Similarly, if you take the same initially vacuum-filled parallel plate capacitor and insert a dielectric of thickness d and area $A/2$ then this can be treated as two capacitors in parallel:

$$C' = C_1 + C_2 = \frac{\varepsilon_0 (A/2)}{d} + \kappa\frac{\varepsilon_0 (A/2)}{d} = \frac{\varepsilon_0 A}{d}\cdot\frac{1}{2}(1+\kappa) = C\cdot\frac{1}{2}(1+\kappa)$$

Note that $C' > C$ for both cases.

2.4 Energy Density

Electric and magnetic fields, since they are made of photons, actually contain energy, which is usually measured as an energy per unit volume – the total energy in the field divided by the volume of the region in which the field exists. In particular, the energy per unit volume of the electric field in a dielectric medium is:

$$u_E = \frac{U_E}{v} = \frac{1}{2}\varepsilon_0 E^2 \kappa \tag{71}$$

where U_E is the total energy in the volume v. (Note that in vacuum or air, we just use $\kappa = 1$.) We can relate this to the stored energy U_C in a parallel plate capacitor by:

$$u_E = \frac{U_C}{v} = \frac{Q^2/2C}{Ad} = \frac{Q^2/2}{Ad}\left(\kappa\frac{\varepsilon_0 A}{d}\right)^{-1} = \frac{1}{2}\varepsilon_0\left(\frac{Q/A}{\varepsilon_0}\cdot\frac{1}{\kappa}\right)^2\kappa = \frac{1}{2}\varepsilon_0\left(\frac{\sigma}{\varepsilon_0}\cdot\frac{1}{\kappa}\right)^2\kappa = \frac{1}{2}\varepsilon_0 E^2\kappa \tag{72}$$

2.5 RC Circuits

When a capacitor is *charging*, the charge on the capacitor varies over time as:

$$Q(t) = Q(\infty)\left(1 - e^{-t/\tau}\right) \tag{73}$$

where $Q(\infty)$ is the final charge on the capacitor and this equation assumes an initial charge of zero. For simple circuits (anything you should ever see) the charging time constant is $\tau = RC$ where R is the resistance in series with the capacitor and C is the capacitance of the capacitor. This equation arises from the fact that the rate at which the charge on the capacitor increases is equal to the input current ($dQ/dt = I$). Solving the associated differential equations gives the above relations. For more complicated circuits, even though $\tau = RC$ may not be readily available, the differential equations should still be capable of providing a solution.

Note that the final voltage across the capacitor is not necessarily equal to the voltage across the battery. In particular $V_C(\infty) \neq \mathcal{E}$ if there is a resistor in parallel with the capacitor and another resistor in series with the resistor-capacitor combination.

When a capacitor is *discharging*, the charge on the capacitor varies over time as:

$$Q(t) = Q(0)e^{-t/\tau} \tag{74}$$

The initial charge on the capacitor is $Q(0)$, which is usually equal to $Q(\infty)$ from the charging circuit. The time constant τ may be different between the charging circuit and the discharging circuit.

3 Resistors

3.1 Basics

$$V = IR \tag{75}$$

is the defining equation for the relationship between the voltage across a resistor, the current passing through it, and its resistance. It is also known as Ohm's law. The units of current are Amperes, $1 \text{ A} = 1 \text{ C/s}$. The units of resistance are Ohms, $1\ \Omega = 1 \text{ V/A} = 1 \text{ Nms/C}^2$.

The *instantaneous* power dissipated by a resistor is:

$$P = VI = I^2R = \frac{V^2}{R} \tag{76}$$

It is notable that for DC circuits this *is* the power dissipation. For oscillatory (AC) circuits, we are usually looking for the *average* dissipated power, which is half the maximum dissipated power.

The energy stored in a resistor is:

$$U \equiv 0 \tag{77}$$

Resistors *only* dissipate energy, never store it. They are the only dissipative circuit element.

3.2 Series and Parallel

Two resistors in series have an equivalent resistance:

$$R = R_1 + R_2 \tag{78}$$

($R > R_1$, $R > R_2$). The current passing through the equivalent resistor is $I = I_1 = I_2$ and its voltage is $V = V_1 + V_2$.

Two resistors in parallel have an equivalent resistance:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \tag{79}$$

($R < R_1$, $R < R_2$). The current passing through the equivalent resistor is $I = I_1 + I_2$ and its voltage is $V = V_1 = V_2$.

3.3 Constructing a Resistor

Resistors are made from material that is imperfectly conductive. Such material is described as having a resistivity ρ, which is measured in units of Ωm. It may also be described as having a conductivity $\sigma = 1/\rho$, measured in units of $\mho \mathrm{m}^{-1}$. ($1\ \mho = 1/(1\ \Omega) = 1$ Mho)

The resistance of a chunk of material of length ℓ and cross-sectional area A (that is, the current travels a distance ℓ in crossing the chunk, and the current flow has a cross-section A) is:

$$R = \frac{\rho \ell}{A} \tag{80}$$