# Physics 212 Exam 2 Review Notes

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## Contents

# 1 Magnetostatics and Ampère's Law

## 1.1 Constants

$$\mu_0 = 4\pi \cdot 10^{-7}\ \frac{\text{Tm}}{\text{A}} = 4\pi \cdot 10^{-3}\ \frac{\text{G m}}{\text{A}} = \text{permittivity of free space} \tag{1}$$

Note that $\mu_0$ is just a constant of nature and is unrelated to $\vec{\mu}$ (units are $\text{Cm}^2/\text{s}$), the magnetic moment of a current loop.

## 1.2 Magnetic Force

The force experienced by a point charge $q$ moving at a velocity $\vec{v}$ through a region with electric field $\vec{E}$ and magnetic field $\vec{B}$ (units of $\vec{B}$ are Teslas or Gauss, $1\ \text{T} = 10^4\ \text{G} = 1\ \text{N/Am}$) is:

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \tag{2}$$

Note that the force due to the magnetic field is perpendicular both to the field and to the charge's velocity, so that a magnetic field can never change the speed of a charge, only the direction of its motion. Also, a charge moving parallel to the magnetic field will experience no force at all! A charge moving perpendicular to a constant magnetic field will move in a circle whose radius is determined by setting $qvB$ equal to the centripetal force $mv^2/R$. So

$$R = \frac{mv}{qB} \tag{3}$$

Its period of revolution is determined by the fact that it takes one period for the charge (going at speed $v$) to go a distance of $2\pi R$, the circumference of the circle. So

$$T = \frac{2\pi R}{v} = \frac{2\pi m}{qB} \tag{4}$$

If the charge has an additional velocity component parallel to the magnetic field, then it will move along a helical path where the radius of the helix is determined by the velocity component perpendicular to the field and the rate of "propagation" of the helix is determined by the velocity component parallel to the field.

From the formula for the magnetic force on a single moving charge, the formula for the force on a current may be determined. If we have a segment of wire of length $d\ell$ carrying a current $I$ flowing in a direction $\hat{e}$ through a region of magnetic field $\vec{B}$, then the force on that tiny wire segment is

$$d\vec{F} = I\, d\ell\, \hat{e} \times \vec{B} \tag{5}$$

To find the force on an entire wire loop, we must integrate this around the whole loop.

For a long straight wire of length $\ell$ in direction $\hat{e}$, the total force on the wire is

$$\vec{F} = I\ell\hat{e} \times \vec{B} \tag{6}$$

while the force per unit length on the wire is:

$$\frac{\vec{F}}{\ell} \text{ or } \frac{d\vec{F}}{d\ell} = I\hat{e} \times \vec{B} \tag{7}$$

A short wire segment of length $d\ell$ carrying current $I$ in a direction $\hat{e}$ creats a magnetic field at a point $P$ a distance $\vec{r}$ from it equal to:

$$d\vec{B} = \frac{\mu_0 I\, d\ell\, \hat{e} \times \vec{r}}{4\pi r^3} \tag{8}$$

To find the total magnetic field at $P$ one must integrate this formula over the entire length of the field-generating wire, not forgetting that the vectors $\vec{r}$ and $\hat{e}$ change as you move to different portions of the wire.

A specific example of the use of this equation is the magnetic field of a wire loop oriented in the $xy$ plane and centered at the origin. The above formula can be used to (relatively) easily compute the magnetic field anywhere on the $z$-axis. If the loop has radius $R$ and current $I$, then the field at a distance $z$ from the origin is:

$$\vec{B} = \pm B_z \hat{z} = \pm \left( \frac{\mu_0 I R^2}{2(z^2 + R^2)^{3/2}} \right) \hat{z} \tag{9}$$

The field off the $z$-axis is much harder to find!

### 1.3 Ampère's Law

Magnetic fields can also be computed using Ampère's law. Like Gauss' law, Ampère's law requires a certain amount of symmetry from the problem. There are four different situations where Ampère's law is useful, which will be described in a moment. The Ampère's law equation is:

$$\oint_{\text{loop}} \vec{B} \cdot d\vec{\ell} = \mu_0 I_{\text{pierce}} \tag{10}$$

The left-hand side means that we're integrating around some chosen loop, travelling in a particular direction (*choose* clockwise or counterclockwise and draw in the arrows before integrating). In the integral, we dot the magnetic field at each point on the loop with the differential path vector $d\vec{\ell}$ at that point. The right hand side of the equation says that this integral must be proportional to the total current which *pierces* the loop, meaning that the wire carrying that current goes through the loop from one side to the other. A well-chosen loop will have a piecewise constant magnetic field magnitude (and relative direction), it will pass through the point where we want to find the field, and it will be pierced by some of the current in the problem. Its symmetry will also match the symmetry of the problem.

#### 1.3.1 Cylindrical Symmetry

Any long straight wire, or similar cylindrically symmetric current distribution will generate a magnetic field of the form

$$B = \frac{\mu_0 I_r}{2\pi r} \tag{11}$$

at a point $P$, where $r$ is the distance of $P$ from the axis of the cylinder and $I_r$ is the total current flowing at a radius less than $r$ from the center of the cylinder. The magnetic field will be directed *tangent* to a circle centered on the cylinder axis with radius $r$.

The computation of $I_r$ can be a little tricky, but here's some possible distributions. For a long straight wire with current $I$

$$I_r = I \tag{12}$$

at any radius $r$. For a long solid cylinder of radius $a$ with total current $I$ distributed evenly over its cross-section

$$I_r = I\frac{r^2}{a^2} \text{ for } r < a \tag{13}$$

$$I_r = I \text{ for } r > a \tag{14}$$

For a long solid cylinder with its center cut out (inner radius $a$, outer radius $b$) and total current $I$ distributed evenly over its cross-section:

$$I_r = 0 \text{ for } r < a \tag{15}$$

$$I_r = I\left(\frac{r^2 - a^2}{b^2 - a^2}\right) \text{ for } a < r < b \tag{16}$$

$$I_r = I \text{ for } r > b \tag{17}$$

For a long thin hollow cylinder with radius $a$ and total current $I$ distributed evenly over its surface:

$$I_r = 0 \text{ for } r < a \tag{18}$$

$$I_r = I \text{ for } r > a \tag{19}$$

### 1.3.2 Current Sheets

For a large solid sheet of current (basically, a bunch of wires lying next to each other with current all flowing in the same direction), the magnetic field does not depend on distance from the sheet. It points parallel to the sheet and perpendicular to the current flow, and it flips direction when you go from one side of the sheet to the other side.

If the sheet is oriented in the $yz$ plane, with the current flowing in the $+\hat{z}$ direction, has $N$ wires per unit length, and each wire carries a current $I$, then the total magnetic field at any point is:

$$\vec{B} = +\frac{1}{2}\mu_0 N I \hat{y} \text{ for } x > 0 \tag{20}$$

$$\vec{B} = -\frac{1}{2}\mu_0 N I \hat{y} \text{ for } x < 0 \tag{21}$$

### 1.3.3 Solenoids

A solenoid is a single wire twisted into a long helix. A current $I$ flows along the wire, and there are $N$ turns of the wire per unit length. Everywhere outside the solenoid, the magnetic field *due to that solenoid* is zero. Everywhere inside the solenoid, the magnetic field is constant and points along the solenoid axis. The magnetic field inside the solenoid due to that solenoid is:

$$B = \mu_0 N I \tag{22}$$

Do not confuse this with the magnetic field of a current sheet. The field inside a solenoid is like the field of two current sheets with their ends joined, so it's twice the field of a single current sheet.

### 1.3.4 Toroidal Coils

A toroidal coil is like a solenoid with its ends joined, to make a donut shape. It has a total of $N$ wire turns, and each turn has a current $I$ flowing along it. In the donut hole and outside the donut, there is no magnetic field. At any point in the "cake" of the donut, at a radius $r$ from the center of the donut hole, there is a magnetic field directed along the tangent to a circle centered at the origin and passing through that point:

$$B = \frac{\mu_0 NI}{2\pi r} \tag{23}$$

## 1.4 Torque and Energy for Current Loops

A current loop can be treated as a magnetic dipole (like a bar magnet). The defining feature of a magnetic dipole is its magnetic moment, $\vec{\mu}$, which is in units of $\text{Am}^2$. For a current loop with $N$ turns, each turn carrying a current $I$ and having area $A$, the magnetic moment is

$$\vec{\mu} = NI\vec{A} \tag{24}$$

The area vector $\vec{A}$ has length equal to the loop area ($A$) and is directed perpendicular to the loop, along the direction of the magnetic field generated by the loop at its center.

A magnetic dipole with moment $\vec{\mu}$ experiences a no net magnetic force when placed in an external magnetic field, but it *will* experience a torque (rotational force, units are Joules) due to the external field:

$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{25}$$

Note that the magnetic field generated by the current flowing in the loop is ignored in these calculations, since it produces no net torque on the loop. (Otherwise, a simple current flowing in a wire loop would rotate the loop all by itself!) Also note that the size of the torque is maximum when $\vec{\mu} \perp \vec{B}$ and minimum (zero) when $\vec{\mu} \parallel \vec{B}$.

Finally, a magnetic dipole with moment $\vec{\mu}$ in an external magnetic field $\vec{B}$ will have a magnetic potential energy equal to:

$$U = -\vec{\mu} \cdot \vec{B} \tag{26}$$

Note that $U$ is maximized when $\vec{\mu}$ and $\vec{B}$ are *antiparallel* (pointing in opposite directions), minimized when $\vec{\mu} \parallel \vec{B}$, and zero when $\vec{\mu} \perp \vec{B}$. The potential energy is maximized (or minimized) when the torque is zero, and the torque has its maximum magnitude when the potential energy is zero.

Recall that a potential energy maximum is an unstable equilibrium, that is, if the loop is nudged a little it will continue to rotate further away from the original orientation. A potential energy minimum is a stable equilibrium — when slightly nudged it will return to its original position. For magnetic moment problems, $U = 0$ is neither the maximum nor minimum value of the potential, and so is not an equilibrium point.

### 1.5 Stored Energy in the Magnetic Field

Like an electric field (see the exam 1 notes) a magnetic field in a region of space contains energy. Specifically, the energy density (energy per unit volume) in a magnetic field is:

$$u_B = \frac{U_B}{v} = \frac{B^2}{2\mu_0} \tag{27}$$

where $U_B$ is the total energy in a volume $v$ filled with a constant magnetic field of amplitude $B$.

Since a solenoid has a constant magnetic field inside it, related to its current by $B = \mu_0 NI$, if the solenoid has length $\ell$ and radius $a$, and $N$ turns per unit length, then the total stored energy in the solenoid when it carries current $I$ is:

$$U_B = u_B v = \left(\frac{B^2}{2\mu_0}\right)\left(\pi a^2 \ell\right) = \left(\frac{(\mu_0 NI)^2}{2\mu_0}\right)\left(\pi a^2 \ell\right) \tag{28}$$

### 1.6 Magnetic Flux

Similar to electrical flux, we can define the magnetic flux through a surface by:

$$\Phi_B = \int_A \vec{B} \cdot d\vec{A} \tag{29}$$

Assuming no magnetic monopoles (ie. no magnetic "charges" analogous to the electric charge) the magnetic flux through any *closed* surface is zero. As a consequence of this, magnetic field lines always form closed loops, while electric field lines *cannot* form simple closed loops. However, the magnetic flux through a non-closed surface (for example, through the disk-shaped surface delineated by a wire loop) is still a useful quantity, as we will see below.

## 2 Inductors, Part I: Induction Into the World of Inductors ;)

### 2.1 Basics

An inductor is essentially the same thing as a solenoid. Its electrical properties can be characterized by a single parameter, called the inductance, which is defined as the magnetic flux through the solenoid divided by the current $I$ which is flowing in it. Given an inductor of length $\ell$ and radius $a$ with $N$ turns per unit length, the inductance is:

$$L \equiv \frac{\Phi_B}{I} = \frac{\vec{B} \cdot \vec{A}}{I} = \frac{(\mu_0 NI)((N \cdot \ell)(\pi a^2))}{I} = \pi \mu_0 N^2 \ell a^2 \tag{30}$$

Note that $A = N\ell \cdot \pi a^2$ because we need the *total* area of all the wire loops through which the magnetic flux is passing. Each loop has an area $\pi a^2$ and there are $N\ell$ loops in the solenoid/inductor.

The potential difference across an inductor, as a circuit element, is

$$V = L\frac{dI}{dt} \tag{31}$$

### 2.2 Series and Parallel

Two inductors in series must have the same current passing through them, $I_1 = I_2$, and so $dI_1/dt = dI_2/dt$. So they can be replaced by an equivalent inductor with total inductance

$$L = L_1 + L_2 \tag{32}$$

($L > L_1$, $L > L_2$) The current in the equivalent inductor will be $I = I_1 = I_2$, so that $dI/dt = dI_1/dt = dI_2/dt$ and $V = V_1 + V_2$.

Two inductors in parallel must have the same voltage difference across them, $V_1 = V_2$, and they may be replaced with an equivalent inductor whose inductance is

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} \tag{33}$$

($L < L_1$, $L < L_2$) The current in the equivalent inductor will be $I = I_1 + I_2$, so that $dI/dt = dI_1/dt + dI_2/dt$ and $V = V_1 = V_2$.

### 2.3 Energy and Power

An inductor never dissipates power, so:

$$P_L \equiv 0 \tag{34}$$

The total energy stored in an inductor at a given moment is related to the current flowing through it at that moment by:

$$U_L = \frac{1}{2}LI^2 = \frac{1}{2}\Phi_B I = \frac{\Phi_B^2}{2L} \tag{35}$$

Note that the stored energy in an inductor (solenoid) is also related to the magnetic field inside it, as discussed in the section on magnetic field.

## 3 Faraday's Law

### 3.1 Basics

The fact that the voltage across an inductor is related to the rate of *change* of the current passing through it suggests that there is some relationship between the changing magnetic field inside the inductor and the potential difference between its ends. The mathematical description of how a changing magnetic field creates an electrical field (and therefore a potential difference, or "electromotive force" (EMF)) is called Faraday's law (named after the English dude who invented the electric motor).

Specifically, Faraday's law states that a changing magnetic field generates an EMF around a closed wire loop based on the magnetic flux passing through said loop. If the loop has an area $A$, then recall that the magnetic flux through the loop at a particular time $t$ is $\Phi_B(t) = \int_A \vec{B}(t) \cdot d\vec{A}$. Then the EMF which drives a current to flow around the loop will be:

$$\mathcal{E}(t) = -\frac{d\Phi_B(t)}{dt} \tag{36}$$

If the area of the loop is constant and the plane of the loop is perpendicular to the (varying) magnetic field, then:

$$\mathcal{E}(t) = -A\frac{dB(t)}{dt} \tag{37}$$

Otherwise, you must carefully analyze what the actual magnetic flux through the loop is *as a function of time* and then take the time derivative.

The current which flows in the loop will be:

$$I(t) = \frac{\mathcal{E}(t)}{R} \tag{38}$$

where $R$ is the *total* resistance of the wire loop. Note that if the wire loop has $N$ coils, then its area is $N \cdot A_1$ where $A_1$ is the area of one coil, and to find the current, you must similarly use the resistance of all $N$ coils.

The sign convention for this equation is a little painful to contemplate. But the trick to remember is that the EMF (and the induced current flow) will circulate around the loop in such a direction that magnetic field generated by the induced current exactly opposes the *change* in the external magnetic field. If the external magnetic field points in the positive $\hat{z}$ direction and its magnitude is *decreasing* then the induced current and EMF will circulate so as to create an induced magnetic field also pointing in the $+\hat{z}$ direction.

Another way to compute the circulation direction for the EMF and current is to actually carefully work through the signs in the Faraday's law equation. You have to start by picking a direction for the area vector, so let's assume for simplicity's sake that the loop is in the $xy$ plane. In this case, the area vector points either in the $+\hat{z}$ or $-\hat{z}$ direction. As viewed from the $+\hat{z}$ side, an area vector in the $+\hat{z}$ direction indicates that the current and EMF circulate counterclockwise, while an area vector in the $-\hat{z}$ direction indicates that the current and EMF circulate clockwise. But since we don't initially know the circulation direction of the current, we cannot be sure which of these directions for the area vector will be correct. All you can really do is pick one and be consistent. If the direction you have chosen is opposite the actual direction, then you will discover that your computed values for the EMF and current are negative (but the magnitudes will still be correct).

This method requires extreme care with the signs. You may pick up minus signs from the $\vec{A} \cdot \vec{B}$ dot product and from the $d\Phi_B(t)/dt$ time derivative, and there will always be a minus sign in the definition of the EMF, $\mathcal{E} = -d\Phi_B(t)/dt$, due to Lenz' law.

Once a current flows in the loop, it can experience a torque due to the external magnetic field as described above. (The induced magnetic field is usually neglected in this computation, since it causes no net torque on the loop.) It will also have a magnetic potential energy due to the external magnetic field.

### 3.2 Types of Faraday's Law Problems

There are three basic problems which you will typically be asked to solve. You may have a stationary wire loop through which the magnetic field is changing. In this case:

$$\mathcal{E}(t) = -A\frac{dB(t)}{dt} \tag{39}$$

You may have a loop of constant area moving from a region of zero magnetic field into a region of constant magnetic field. In this case, the flux through the loop is constant as long as it is completely outside or completely inside the field region, so there is no EMF and no current in either of these cases. The loop will only experience an EMF as it is *entering* or *leaving* the magnetic field region, and the EMF will then be:

$$\mathcal{E}(t) = -B\frac{dA(t)}{dt} \tag{40}$$

where $A(t)$ represents the amount of the loop's area that is within the magnetic field region. Note that if the EMF and current flow were clockwise as the loop entered the region, they will be counterclockwise as it leaves, and *vice versa*.

Finally, you may have a loop of constant area in a region of constant magnetic field, where the loop rotates on a pivot (usually one of its edges). Then the loop area and the magnetic field are constant, but their relative direction changes. So when the loop's area vector makes an angle $\theta$ with the magnetic field, then the magnetic flux through the loop is:

$$\Phi_B = AB\cos(\theta) \tag{41}$$

and the EMF around the loop is:

$$\mathcal{E}(t) = AB\frac{d\theta}{dt}\sin(\theta) \tag{42}$$

Usually, the loop will rotate at a constant angular velocity $\omega$, so that $\theta(t) = \omega t + \theta_0$ and:

$$\mathcal{E}(t) = AB\omega\sin(\omega t + \theta_0) \tag{43}$$

(Recall that angular velocity is mathematically very similar to linear velocity. For an object starting at a linear position $x_0$ and travelling at a constant linear velocity $v$, the position at time $t$ is $x = x_0 + vt$. For an object starting at an angle $\theta_0$ and rotating at a constant angular velocity $\omega$, the angle at time $t$ is $\theta = \theta_0 + \omega t$.)

There is one final problem, which I don't know if you'll get or not, where the magnetic field is constant, but the loop area is changing (eg. one side is a bar sitting on metal rails, moving along the rails). For this case the EMF is:

$$\mathcal{E}(t) = -B\frac{dA(t)}{dt} \tag{44}$$

### 3.3 An Additional Point

Just because the flux is zero, it does not mean the EMF is zero, and conversely, just because the EMF is zero, it does not mean that the flux is zero. The EMF is related to the *change* over time of the flux, so if the flux is zero but increasing, the EMF will be nonzero. Conversely, if the EMF is zero, the flux may be non-zero but constant or it may be instantaneously at its maximum or minimum value.

# 4 Inductors, Part II: DC RL Circuits

An inductor resists having its current changed. So if an inductor initially has no current flowing through it, when we connect it up to a potential difference, no current will immediately be able to flow through it (or anything in series with it). But since $V = L(dI/dt)$, the current will slowly start to increase, increasing the stored energy in the inductor ($U_L = LI^2/2$).

If the inductor is simply in a circuit with a battery, theoretically the current through the inductor will simply increase at a constant rate until all the energy stored in the battery has been transferred to the inductor (and the voltage difference across both is zero, because the battery has no energy and the current through the inductor is constant). So usually, we put a resistor in series with the inductor, thus limiting the maximum current flow through the inductor.

The theory in this case is that we let the circuit evolve over time until it reaches a steady state. In the steady state, the current through the inductor is constant and so $dI/dt = 0$ (although $I$ may or may not be zero), and the voltage across the inductor is also zero. (It acts like a wire.) So if we simply put a resistor in series with the inductor, in the steady state all of the battery voltage will be across the resistor, and the current through the resistor (and inductor) will simply be $I = \mathcal{E}/R$.

As the inductor is charging up to this maximum current, we can compute the current through it at any time using a formula similar to that for a charging capacitor:

$$I(t) = I(\infty)\left(1 - e^{-t/\tau}\right) \tag{45}$$

The $I(\infty)$ is simply the final current through the inductor, while the time constant $\tau$ is:

$$\tau = \frac{L}{R} \tag{46}$$

where $R$ is the resistance in series with the inductor $L$.

When we take the battery out of the circuit and discharge the inductor, the current through the inductor will vary as:

$$I(t) = I(0)e^{-t/\tau} \tag{47}$$

Usually, $I(0)$ will be $I(\infty)$ from the charging circuit, but depending on how the discharge circuit is constructed, the time constant $\tau$ may be different.

# 5 Displacement Current

This is kind of a difficult (and lightly-discussed) subject, so I don't know if it will come up on the exam, but just in case...

Just as a changing magnetic field creates an electric field (Faraday's law), a changing electric field can also create a magnetic field, *as if* there was a current flowing nearby. This imaginary current is called the "displacement current". If you want to find the induced magnetic field at a given point $P$ due to a changing electric field then these are the steps you follow:

First draw an Ampèrian loop which passes through $P$, is perpendicular to the change in the electric field, and shares the symmetry of the charge/current distribution.

Next, compute the electric flux passing through the Gaussian surface enclosed by that loop, using Gauss' law, as a function of time.

The displacement current is defined as:

$$I_D = \varepsilon_0 \frac{d\Phi_E}{dt} \tag{48}$$

This current will be the total piercing current for your Ampèrian loop, and it will flow in the direction of the *change* in the electrical field. You can then use it to compute the magnetic field via Ampère's law.

The canonical example is a charging parallel-plate capacitor with circular plates. The plates each have radius $a$, and a current $I$ is going into the left plate and out of the right one. So the charge on the left plate at any given time is $Q(t)$, where $I(t) = dQ(t)/dt$. The right plate, of course, has a charge of $-Q(t)$. The electric field between them can be approximated by treating the capacitor plates as infinite parallel plates, so $\vec{E(t)} = \hat{x}(Q(t)/A_{\text{plate}})/\varepsilon_0 = \hat{x}Q(t)/(\pi\varepsilon_0 a^2)$.

We want to find the induced magnetic field at a distance $r$ from the center axis of the plates. So we draw an Ampèrian loop of radius $r$ aligned in the $yz$ plane and centered along the plate axis (the $x$-axis). The Gaussian surface defined by this Ampèrian loop is a disk of radius $r$, also centered at the plate axis. So the electric flux through this Gaussian surface is $\Phi_E(t) = E(t)A_{\text{loop}} = (Q(t)/\pi\varepsilon_0 a^2)(\pi r^2) = Q(t)r^2/\varepsilon_0 a^2$. Thus the displacement current is $I_D = \varepsilon_0 \cdot d\Phi_B(t)/dt = (r^2/a^2)I$. Since the electric field is increasing in magnitude and pointing right, the displacement current also flows right.

So the magnetic field at a radius $r$ from the center of the plates is simply that of a current $I_D$ flowing to the right along the $x$ axis. It will therefore satisfy Ampère's law: $\int_{\text{loop}} \vec{B} \cdot d\vec{l} = \mu_0 I_D$, where the loop has radius $r$ and the magnetic field is parallel to the loop at every point. Thus the magnetic field is:

$$B = \mu_0 I \frac{r^2}{a^2} \div (2\pi r) = \frac{\mu_0 I r}{2\pi a^2} \tag{49}$$